Reflection Coefficient of a Conducting Sphere on the Broad Wall of a Rectangular Waveguide

JOHN D. CASHMAN AND HARRY E. GREEN, SENIOR MEMBER, IEEE

Abstract —**Small spherical objects have been found useful as impedance matching elements in rectangular waveguides. In this paper, we develop a formula for the reflection coefficient produced by a conducting spherical ball in contact with the broad wall of a rectangular waveguide. The solution involves replacement of the obstacle by equivalent electric and magnetic dipoles but employs no ad hoc assumptions to determine the dipole moments and to this extent is exact. The theory is found to yield good results for all balls likely to be practical as impedance matching elements.**

I. INTRODUCTION

THE USEFULNESS of a conducting spherical obstacle as a means for matching out small reflections in a rectangular waveguide propagating the fundamental (H_{10}) mode was pointed out some time ago by Somlo and Hollway [1]. Provided that the ball at least has a ferrous core, it can be positioned from outside the waveguide by a magnet, which is often convenient in experimental work. These authors noted that the equivalent shunt susceptance produced by such an obstacle is capacitive and that, for a ball positioned on the center of one broad wall, the magnitude of the reflection coefficient which it generates remains constant within ten percent over the entire waveguide band.

Based on the results of a number of measurements at X band in WG16 (RG52/U), Somlo and Hollway were able to derive an empirical formula for the reflection coefficient generated by a conducting ball placed centrally on the broad wall of the guide, as shown in Fig. 1. For $0 < r/b < 0.47$, they concluded that

$$|\Gamma| = \frac{46.4(r/b)^3}{1+89.6(r/b)^4} \tag{1}$$

which, for $r/b < 0.15$, may be approximated with an error not exceeding five percent by the simpler form

$$|\Gamma| = 46.4(r/b)^3. \tag{2}$$

Somlo and Hollway went on to remark that, for the behavior they had observed, "no theoretical basis...has been found."

Manuscript received June 15, 1983; revised January 24, 1984.
The authors are with the University of New South Wales, Royal Military College, Department of Electrical and Electronic Engineering, Duntroon, A.C T., 2600 Canberra, Australia.

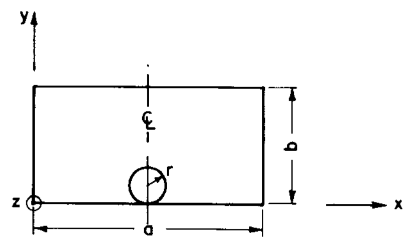

Fig. 1. Waveguide, dimensions $a \times b$, with conducting sphere, radius r touching the center of the broad wall. Coordinate axes x and y are along broad and narrow walls, respectively. Propagation is in the z direction.

II. CRITIQUE OF EXISTING SOLUTIONS

The behavior of a small obstacle in a waveguide may be determined by replacing it with equivalent electric and magnetic dipoles, the strength of which depend on the exciting fields and the electric and magnetic polarisabilities of the obstacle. These polarisabilities are dyadics, the elements in which are proportional to the volume of the obstacle, the proportionality being determined by the detailed geometry of the obstacle. Thus, it is not surprising that the magnitude of the reflection coefficient should have the form shown in (2). Additionally, we shall soon see that the capacitive nature of the susceptance is not a matter for surprise.

Referred to a set of principal axes which diagonalise the dyadic, for some simple geometries it is possible to determine these polarisabilities by appropriate analysis. An isolated ellipsoid, of which an isolated sphere is but a special case, is an example. When placed in a waveguide, the sphere is no longer isolated but is part of an infinite, planar array of which all the other members are its images in the waveguide walls. It can be assumed safely that its polarisability remains the same as that of an isolated sphere only if the elements in the array are sufficiently separated. When this is not the case, two effects are in evidence. The first is that the exciting field will be modified by contributions due to the images, but the more serious is that the charge and current distributions induced on the obstacle may no longer be similar to those experienced in isolation.

When the obstacle is not near any of the waveguide walls, only the first is of importance and may be taken into

0018-9480/84/0600-0582$01.00 ©1984 IEEE

account with sufficient accuracy by quasistatic arguments, the polarisabilities used being those of an isolated obstacle. This can be expected to give an acceptable result for a sphere suspended on the longitudinal axis of symmetry of the waveguide, but for a sphere placed on the broad wall, this can involve discrepancies between theory and measurement in the order of 2:1.

A theoretical solution to this problem which gives better agreement with experiments has been published [2]. It relies on replacing the sphere lying on the face of the waveguide by an equivalent half ellipsoid. That the reflection coefficient is capacitive and proportional to obstacle volume is then readily determined. However, that this must be the case for any small obstacle, whatever its shape, is easy to see; the ball acts as a perturbation to remove energy from the field in the region where the electric energy density is the greater and, to the extent that this field is uniform, the amount of energy removed is proportional to the obstacle volume [3]. Given that determination of the equivalent ellipsoid depends on ad hoc assumptions, the nature of which would not be evident without knowledge of the experimental result with which it is desired to achieve correspondence, it is difficult to see that this solution adds much to fundamental understanding of the problem.

More recently, a finite-element solution has appeared [4]. This solution avoids the approximations and ad hoc assumptions of [2], but leads only to a numerical result. The present result takes form of an analytic formula which thereby provides a greater physical insight into the problem.

III. Theoretical Development

A. Formal Solution

In this paper, we shall adopt a different approach. We shall assume, as has already been made evident, that the ball lies on the broad wall of the waveguide but with the additional restriction that it is not too close to the narrow walls. We then image the ball in the wall on which it lies so that we are left to analyze the effect of a dumbbell-shaped obstacle consisting of two osculating spheres which lies along the horizontal centerline of a rectangular waveguide having cross-sectional dimensions $a \times B$, where $B = 2b$. Fig. 2 shows the result. If we determine the electric and magnetic polarisability dyadics for this dumbbell, we can, provided that it does not come too close to the side walls, estimate the modifying effect on the exciting field of the images by the usual quasistatic arguments.

Determination of the elements of the polarisability dyadics is to be the subject of a separate paper by Cashman [5], and here we shall simply use his results in the endeavor outlined above. We begin by outlining briefly the formal solution to the problem, full details of which can be found in standard texts [6]. We assume that the long axis of the dumbbell lies in the plane $z = 0$. Then, the electric and magnetic fields incident on the obstacle are

$$\bar{E}_t = V^+ \bar{e}_{10} e^{-j\beta_{10} z} \tag{3}$$

$$\bar{H}_t = V^+ \left(\bar{h}_{10} + \bar{h}_{z10}\right) e^{-j\beta_{10} z} \tag{4}$$

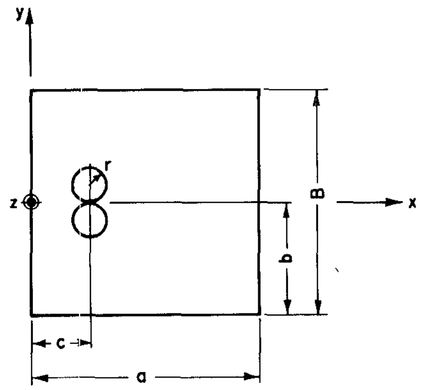

Fig. 2. Doubled waveguide containing a dumbbell obstacle representing the original guide and the sphere and images in the xz plane. New dimension $B = 2b$. Sphere and its image have been moved off center to distance c from left-hand wall.

where

$$\bar{e}_{10} = -\hat{y} k_0 Z_0 \left(\frac{2}{aBk_0 Z_0 \beta_{10}} \right)^{1/2} \sin\left(\frac{\pi x}{a}\right) \tag{5}$$

$$\bar{h}_{10} = \hat{x} \beta_{10} \left(\frac{2}{aBk_0 Z_0 \beta_{10}} \right)^{1/2} \sin\left(\frac{\pi x}{a}\right) \tag{6}$$

$$\bar{h}_{z10} = -\hat{z} j \frac{\pi}{a} \left(\frac{2}{aBk_0 Z_0 \beta_{10}} \right)^{1/2} \cos\left(\frac{\pi x}{a}\right) \tag{7}$$

are the normal mode functions for the H_{10} mode in an $a \times B$ rectangular waveguide, and in which

$\beta_{10} = k_0\sqrt{1 - (\lambda_0/\lambda_c)^2}$ the guide wavenumber,

$k_0 = \dfrac{2\pi}{\lambda_0}$ the free-space wavenumber,

λ_0 the free space wavelength,

$\lambda_c = 2a$ the cutoff wavelength, and

Z_0 the characteristic impedance of space.

The incident fields will induce in the dumbbell electric and magnetic dipole moments

$$\bar{P} = \epsilon_0 \bar{\bar{A}}_e \cdot \bar{E}_t |_{(x,y,z)=(c,0,0)} \tag{8}$$

$$\bar{M} = \bar{\bar{A}}_m \cdot \bar{H}_t |_{(x,y,z)=(c,0,0)} \tag{9}$$

where $\bar{\bar{A}}_e$, $\bar{\bar{A}}_m$ are dyadics to be determined which depend on the electric and magnetic polarisabilities of the dumbbell and the disposition of its images. By symmetry, it is clear that the principal axes of the dyadic must lie along the x, y, and z axes, so that we may write

$$\bar{\bar{A}}_e = A_{exx}\hat{x}\hat{x} + A_{eyy}\hat{y}\hat{y} + A_{ezz}\hat{z}\hat{z} \tag{10}$$

$$\bar{\bar{A}}_m = A_{mxx}\hat{x}\hat{x} + A_{myy}\hat{y}\hat{y} + A_{mzz}\hat{z}\hat{z}. \tag{11}$$

In terms of these components, (8) and (9) with (3) and (4) give

$$\bar{P} = V^+ \epsilon_0 A_{eyy} \bar{e}_{10} |_{(x,y)=(c,0)} \tag{12}$$

$$\bar{M} = V^+ \left(A_{mxx} \bar{h}_{10} + A_{mzz} \bar{h}_{z10}\right) |_{(x,y)=(c,0)}. \tag{13}$$

In $z<0$, these generate scattered fields

$$\bar{E}_s = V^- \bar{e}_{10} e^{j\beta_{10} z} \tag{14}$$

$$\bar{H}_s = V^- \left(-\bar{h}_{10} + \bar{h}_{z10}\right) e^{j\beta_{10} z}. \tag{15}$$

Application of the Lorentz reciprocity theorem then serves to show that

$$V^- = \tfrac{1}{2} j\omega \left[\mu_0 \left(\bar{h}_{10} + \bar{h}_{z10}\right) \cdot \bar{M} - \bar{e}_{10} \cdot \bar{P}\right]\Big|_{(x,y)=(c,0)} \tag{16}$$

whence the reflection coefficient

$$\begin{aligned}\Gamma &= \frac{V^-}{V^+} \\ &= \frac{j\omega}{aBk_0 Z_0 \beta_{10}} \left[\mu_0 \left\{\beta_{10}^2 A_{mxx} \sin^2 \frac{\pi c}{a} - \left(\frac{\pi}{a}\right)^2 A_{mzz} \cos^2 \frac{\pi c}{a}\right.\right. \\ &\quad \left. - \epsilon_0 k_0^2 Z_0^2 A_{eyy} \sin^2 \frac{\pi c}{a}\right] \\ &= \frac{jk_0}{2ab} \left[\left(\frac{\beta_{10}}{k_0} A_{mxx} - \frac{k_0}{\beta_{10}} A_{eyy}\right) \sin^2 \frac{\pi c}{a}\right. \\ &\quad \left. - \frac{1}{k_0 \beta_{10}} \left(\frac{\pi}{a}\right)^2 A_{mzz} \cos^2 \frac{\pi c}{a}\right].\end{aligned} \tag{17}$$

For a ball on the center of the broad wall of the waveguide ($c = \frac{1}{2}a$), this simplifies to

$$\Gamma = \frac{jk_0}{2ab} \left[\frac{\beta_{10}}{k_0} A_{mxx} - \frac{k_0}{\beta_{10}} A_{eyy}\right]. \tag{18}$$

This completes the formal solution of the problem. It remains only to determine certain components of the $\bar{\bar{A}}_e$, $\bar{\bar{A}}_m$.

B. Determination of the Polarisability Dyadics

Provided that the dumbbell and its images are sufficiently separated, the presence of the images does not cause current and charge patterns to be induced which differ substantially from those which occur in isolation, and it has been shown that the elements of the $\bar{\bar{A}}_e$, $\bar{\bar{A}}_m$ dyadics may be written as

$$A_{e,muu} = \frac{\alpha_{e,muu}}{1 - C_{e,mu} \alpha_{e,muu}} \tag{19}$$

where $u = x, y, z$ and $\alpha_{e,muu}$ are the corresponding elements of the electric or magnetic polarisability dyadic for an isolated dumbbell. $C_{e,mu}$ is an interaction constant, considered again below, which takes into account that the exciting field is modified by the presence of the images.

The dyadic elements α_{eyy}, α_{mxx}, and α_{mzz} have been determined by Cashman [5] using a quasistatic approach justified by the fact that the dumbbell is electrically small. Under an imposed field, a charge separation (in the case of a magnetic field, a separation of hypothetical magnetic charge) takes place, determined by appropriate boundary conditions. The boundary condition is expressed as an integral equation which is solved numerically for the charge distribution. The dipole moment is found from the charge distribution and the polarisability elements $\alpha_{e,muu}$ then

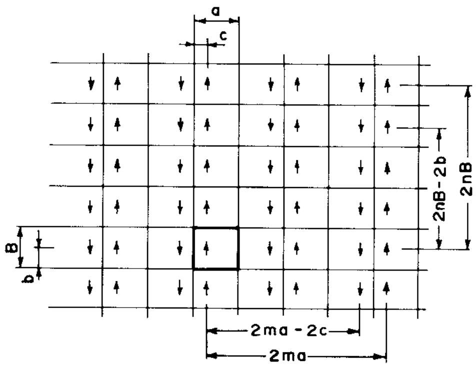

Fig. 3 y-directed electric dipole within the doubled waveguide (shown in heavy lines) and its images in the conducting walls.

follow. The results are

$$\alpha_{eyy} = 59.65 r^3 \tag{20}$$

$$\alpha_{mxx} = \alpha_{mzz} = -14.02 r^3. \tag{21}$$

It is interesting to compare these results with those obtained by simply doubling up the corresponding polarisabilities of an isolated sphere. If we did this, we would get $\alpha_{eyy} = 25.13 r^3$, $\alpha_{mxx} = \alpha_{mzz} = -12.57 r^3$. To do so is to fail to take account of the changes in the charge and current distributions on the ball which result from its immediately adjacent image in the broad wall on which it rests. The resulting errors are seen to be very significant.

To determine the interaction constants, we follow the quasistatic theory of Collin [6]. Consider first the y-directed electric dipole moment. This images positively in the broad walls of the guide (to which it is perpendicular) and negatively in the narrow (to which it is parallel). The result is the alternate lines of positively and negatively directed dipoles shown in Fig. 3 which form the basis to determine the component of the polarizing field at the dumbbell due to its images. For this case, we find that

$$C_{ey} = \frac{1}{4\pi} \left[S_1 + S_2 - S_3 - S_4\right] \tag{22}$$

where

$$S_1 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{2(2nB)^2 - (2ma)^2}{\left[(2nB)^2 + (2ma)^2\right]^{5/2}} \tag{23}$$

$$S_2 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2(2nB-2b)^2 - (2ma)^2}{\left[(2nB-2b)^2 + (2ma)^2\right]^{5/2}} \tag{24}$$

$$S_3 = \sum_{n=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2(2nB^2 - (2ma-2c)^2}{\left[(2nB)^2 + (2ma-2c)^2\right]^{5/2}} \tag{25}$$

$$S_4 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2(2nB-2b)^2 - (2ma-2c)^2}{\left[(2nB-2b)^2 + (2ma-2c)^2\right]^{5/2}}. \tag{26}$$

The superscript prime on the first of the summations indicates that the $n = 0$ term is to be omitted. These series are readily summed to convergence by computer.

Turning now to the magnetic dipoles and considering the x-directed dipoles first, we observe that we have a situation exactly analogous to that just considered. This follows because the dipole images positively in the walls to which it is parallel and negatively in those to which it is perpendicular. Hence, we have at once that $C_{mx} = C_{ey}$. This leaves only the z-directed magnetic dipoles to consider. These lie parallel to the broad and narrow walls of the guide and so all the images are positive. Hence, we have

$$C_{mz} = \frac{1}{4\pi}[S_1 + S_2 + S_3 + S_4]. \tag{27}$$

Note that, because they all lie on $z = 0$, there is no interaction between the x- and z-directed sets of magnetic dipoles.

Inserting these results into (18), we see that the reflection coefficient is indeed capacitive and to the lowest order in r is proportional to r^3. However, if our result is expanded as a power series in r, the next term will involve r^6. This is to be contrasted with Somlo and Hollway's empirical formula which, if rewritten as a power series, has a second term with an r^7 dependence. This may perhaps be reconciled by noting that the empirical result applies to a range of ball sizes extending beyond that for which it is valid simply to replace the ball by electric and magnetic dipole moments located at its center, the higher order multipoles needing to be considered in a complete solution.

IV. Some Results

Using our formula (18) we have computed $|\Gamma|$ as a function of ball diameter at band center (9.6 GHz) in WG16 (RG52/U) for which $a = 22.86$mm (0.900 in), $b = 10.16$mm (0.400 in). It is shown in Fig. 4, where it is compared with the results of Somlo and Hollway, both their measurements and their empirical formula. The agreement is seen to be very good for any ball below 5mm in diameter or, to put it another way, for any reflection coefficient with a magnitude below 0.5. Given the intended use of the device—for Vernier matching—it is seen that the theory gives an adequate description of its performance over all practical ball sizes.

In Fig. 5, we present for 3-mm and 4-mm-diam balls located on the center of the broad wall of WG16 a portrayal of $|\Gamma|$ against frequency in the range 8–11 GHz. The result supports Somlo and Hollway's assertion of the relative insensitivity of the reflection coefficient to frequency. Lastly, also for 3-mm and 4-mm-diam balls but this time at band center, we present in Fig. 6 the result of moving the ball transversely in the guide over the range $a/4 < c < 3a/4$, this being about as close to the narrow wall as it is possible to bring the ball and still use polarisabilities calculated for an isolated dumbbell. It is observed that, rather than having to replace the ball in order to vary the reflection coefficient, useful variation can be achieved simply by moving it off guide center. The reflection coefficient falls off because the exciting fields, and therefore the dipole moments which they produce, are generally weaker off center. This does not, of course, apply to the z-directed magnetic dipole moment, which is strongest at the edge of the guide, but the effect of which in the expression for $|\Gamma|$ is subtractive from that due to the x-directed magnetic dipole.

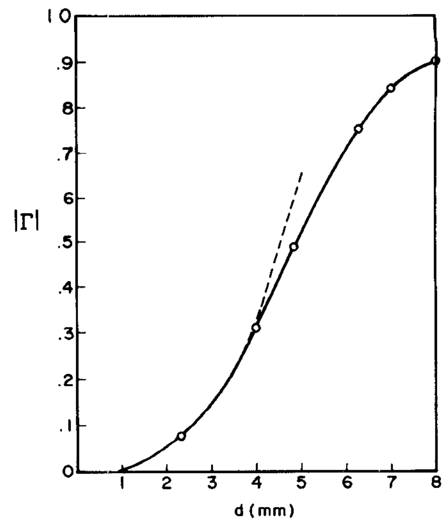

Fig. 4. Reflection coefficient magnitude $|\Gamma|$ as a function of ball diameter d. Circles are experimental points of Somlo and Hollway. Full curve is (1). Dashed curve is present theoretical result (2) with ball on the center of the broad face of WG16 guide. Frequency is 9.6 GHz.

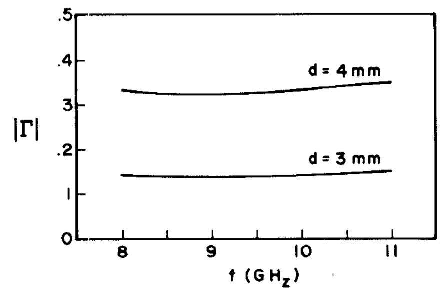

Fig. 5. $|\Gamma|$ as function of frequency for two-ball diam d. Ball ix on center of the broad face of WG16 guide.

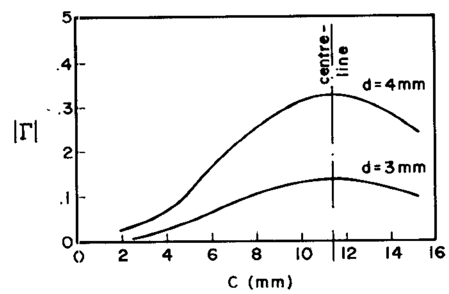

Fig. 6. $|\Gamma|$ as function of distance c from narrow face of WG16 guide for two-ball diam. Frequency is 9.6 GHz.

V. Conclusion

This paper has considered the problem of computing the reflection coefficient produced by a conducting ball when in contact with the broad wall of a rectangular waveguide, a device which has been proposed as a convenient matching element. It has been shown that the reflection coefficient is capacitive, to the lowest order proportional to the cube of the ball radius, and, for a given size ball, is relatively insensitive to frequency. These conclusions confirm earlier empirical work by Somlo and Hollway. The solution is based on the replacement of the ball by a set of electric and magnetic dipole moments located at it center. Strictly, it is therefore applicable only to small balls, but comparison with measurement shows that it is adequate up to the largest likely to be used in practice as a matching element.

Acknowledgment

The authors would like to thank P. I. Somlo of the Australian National Measurement Laboratory for drawing their attention to the problem.

References

[1] P. I. Somlo and D. L. Hollway, "Conductive contacting spheres on the centre of the broad wall of rectangular waveguides," *Electron. Lett.*, vol. 8, no. 20, pp. 507–508, Oct. 1972.

[2] J. H. Hinken, "Conducting spheres in rectangular waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 711–714, July 1980.

[3] J. L. Altman, *Microwave Circuits*, 1st ed. New York: Van Nostrand, 1964.

[4] J. P. Webb and R. L. Ferrari, "Finite-element solution of three-dimensional electromagnetic problems," *IEE Proc.*, vol. 130, Pt. H., no. 2, pp. 153–159, Mar. 1983.

[5] J. D. Cashman, "Electric and magnetic polarisabilities of contiguous conducting spheres," to be published in *J. Elec. Electron Eng.* (Australia).

[6] R. E. Collin, *Field Theory of Guided Waves*. New York: McGraw-Hill, 1960.

John D. Cashman was born in Sydney, Australia, in 1940. He received the M.Sc. degree in physics from the Australian National University and the Ph.D. degree in electrical engineering from the University of New South Wales.

He has been a Lecturer, then Senior Lecturer, in the Faculty of Military Studies of the University of New South Wales since 1967. In 1973–74, he took study leave for nine months with the ElectroScience Laboratory of the Ohio State University, Columbus, and in 1978 a study leave for seven months with the Ecole Supérieure d'Eléctricité at Gif-sur-Yvette, near Paris. His working interests include the application of the Method of Moments and the Geometrical Theory of Diffraction to scattering problems.

Harry E. Green (SM'71) was born in Adelaide, South Australia, on February 11, 1935. He obtained Bachelors and Masters degrees in electrical engineering from the University of Adelaide, and the Ph.D. degree from Ohio State University, Columbus.

After a number of years in the Australian Defence Scientific Service, he joined the University of New South Wales in 1972 to take the Chair in Electrical Engineering in the Faculty of Military Studies, Royal Military College, Duntroon. His professional interests are in electromagnetic theory with applications in antennas, diffraction, and microwave circuits. In 1977, he was awarded the John Madsen Medal by the Institution of Engineers, Australia, and in 1981 the Norman V. W. Hayes Memorial Medal by the Institution of Radio and Electronics Engineers, Australia.

Professor Green is a Fellow of the Institution of Engineers, Australia, and of the Institution of Radio and Electronics Engineers, Australia, and a member of Sigma Xi. Currently, he is serving on the I.E.E.E. Board of Directors as Director, Region 10 (Asia and the Pacific).